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GLOBAL EXISTENCE AND EXPONENTIAL STABILITY FOR A COUPLED WAVE SYSTEM

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Abstract

Using semi-groups theory, regularity results and a theorem associated to the Lumer Phillips theorem, we prove the existence of global solution for a coupled wave system. Also, using multiplicative techniques and the classic Gearhart theorem, introduced in Liu-Zheng [7], we prove that energy associated to the system decays exponentially to zero when $t \to +\infty$. Here, we give two new and interesting proofs.

Stability for a coupled wave system has been considered in [13], where they used Prüs result [12].

1. Introduction

The study of asymptotic behaviour for dissipative systems is a very productive researching field in partial differential equations. In this way, to obtain rates of decay, some analytic techniques were used by several authors, for example, the Komornik method [5], the Nakao method

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[10, 17, 18, 19], and the energy method [8, 14]. In this paper, we use another analytic technique introduced by Zheng [7], which is also applied to dissipative problems, like [15, 16]. This powerful -and apparently simple-method consist in exploring the dissipative properties of the semi-group C_o associated to the system, through the resolvent of its infinitesimal generator.

Since $E(t)$ the total energy associated to coupled wave system,

$$E(t) = \frac{1}{2}\int_{\Omega}\{|\nabla u|^2 + |u_t|^2 + |\nabla v|^2 + |v_t|^2\}dx,$$

have a non positive derivative, that is $E'(t) \leq 0$, the system is dissipative.

Then, we want to know if $E(t) \to 0$ when $t \to +\infty$, and what is its decay rate?. The answer is affirmative, that is, there exist positive constants C and γ such that

$$E(t) \leq CE(0)e^{-\gamma t} \text{ for every } t > 0.$$

This result extend the one in [9] in the sense that we allow higher dimension spaces and variable friction coefficient.

Remember that if α is a constant, then the stability result is satisfied to the model

$$u_{tt} - u_{xx} + \alpha u_t = 0,$$

we can cite [16]. And also, it is true to the model

$$u_{tt} - u_{xx} + \alpha(x)u_t = 0.$$

It can see, for example, in [3, 4, 19].

Stability for coupled wave system has been considered in [1, 2, 6, 9, 13] among others.

Thus, our main goal is to prove the existence and uniqueness of global solution of a coupled wave system and its exponential stability.

We prove the existence of global solution for a coupled wave system by using semi-groups theory. Here, we give a full proof. Also, using multiplicative techniques and the classic Gearhart theorem, introduced in Liu-Zheng [7], we prove that energy associated to the system decays exponentially to zero when $t \to +\infty$. Here, we give two interesting proofs.

Our paper is organized as follows. In Section 2, we state the preliminary results that we will use. In Section 3, we prove the existence and uniqueness of global solution. In Section 4, we prove the exponential decay of the solution.

2. Preliminaries

To prove existence of solution, we will use a result associated to Lumer Phillips theorem. Here, we state this important result. The proof can be seen in Pazy [11].

Theorem 2.1. *Let A be a linear operator with domain $D(A)$ dense in a Hilbert space. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_o semi-group of contraction in this Hilbert space.*

We know that the problem of providing an estimate to the energy $E(t)$ of the form

$$E(t) \leq CE(0)e^{-wt}, \quad \forall t \geq 0,$$

is equivalent to providing exponential stability for semi-group $S(t)$

$$\|S(t)\| \leq Ce^{-wt}, \quad \forall t \geq 0,$$

we cite Liu-Zheng [7].

A necessary and sufficient condition for a semi-group C_o to be exponentially stable is given by the following result:

Theorem 2.2 (Gearhart). *Let* $(S(t))_{t\geq 0}$ *be a* C_o *semi-group of contraction in a Hilbert space. Then,* $(S(t))_{t\geq 0}$ *is exponentially stable* (*that is*, $\exists M \geq 1$, $\mu > 0$ *such that* $\|S(t)\| \leq Me^{-\mu t}$, $\forall t \geq 0$), *if and only if*

(a) $\rho(A) \supset i\mathbb{R} := \{i\beta, \beta \in \mathbb{R}\}$,

(b) $\lim\sup_{|\beta|\to\infty} \|(i\beta I - A)^{-1}\| < \infty.$

We will, respectively, use Theorems 2.1 and 2.2 to prove existence of solution and exponential stability of a coupled wave system.

3. The Abstract Cauchy Problem and Existence of Solution

Here, we study the following system of coupled wave equations:

$$u_{tt} - \Delta u + \alpha(x)(u_t - v_t) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \tag{3.1}$$

$$v_{tt} - \Delta v - \alpha(x)(u_t - v_t) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \tag{3.2}$$

$$u(x, t) = v(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}^+, \tag{3.3}$$

$$u(x, 0) = u_o(x), \; u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{3.4}$$

$$v(x, 0) = v_o(x), \; v_t(x, 0) = v_1(x), \quad x \in \Omega, \tag{3.5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. Here α is a function such that $\alpha \in W^{1,\infty}(\Omega)$, $\alpha(x) \geq 0$ in Ω and $\int_\Omega \alpha(x)dx = \alpha_0 > 0.$ This means that $\alpha(x)$ can vanish at some points of Ω, but the measure of this support is positive.

To get the energy associated to the system, multiply (3.1) by u_t and integrate on Ω, having

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \{|u_t|^2 + |\nabla u|^2\}dx + \int_\Omega \alpha(x)(u_t - v_t)u_t dx = 0, \tag{3.6}$$

also multiply (3.2) by v_t and integrate on Ω to have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}\{|v_t|^2+|\nabla v|^2\}dx-\int_{\Omega}\alpha(x)(u_t-v_t)v_t dx=0. \tag{3.7}$$

Summing (3.6) with (3.7), we get

$$\frac{\partial}{\partial t}\frac{1}{2}\int_{\Omega}\{|u_t|^2+|v_t|^2+|\nabla u|^2+|\nabla v|^2\}dx+\int_{\Omega}\alpha(x)|u_t-v_t|^2 dx=0. \tag{3.8}$$

Let

$$E(t):=\frac{1}{2}\int_{\Omega}\{|u_t|^2+|v_t|^2+|\nabla u|^2+|\nabla v|^2\}dx,$$

be the energy associated to the system (3.1)-(3.5). Then

$$E'(t)=-\int_{\Omega}\alpha(x)|u_t-v_t|^2 dx,$$

and since $\alpha(x)\geq 0$, we have

$$E'(t)\leq 0,$$

that is, the system is dissipative.

With this $E(t)$ in mind, we introduce the following space:

$$X:=H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega), \tag{3.9}$$

endowed with the norm

$$\|U\|_X:=\int_{\Omega}\{|\nabla u|^2+|\varphi|^2+|\nabla v|^2+|\psi|^2\}dx,$$

for $U=(u,\varphi,v,\psi)^T\in X$.

We remark that X is endowed with the scalar product

$$\begin{aligned}<U_1,U_2>_X:=&<\nabla u_1,\nabla u_2>+<\varphi_1,\varphi_2>\\&+<\nabla v_1,\nabla v_2>+<\psi_1,\psi_2>,\end{aligned} \tag{3.10}$$

where

$$U_i = \begin{pmatrix} u_i \\ \varphi_i \\ v_i \\ \psi_i \end{pmatrix} \in X, \quad \text{para } i = 1, 2,$$

and $< \cdot, \cdot >$ denotes the scalar product in $L^2(\Omega)$. Thus, X is a Hilbert space.

Define

$$\varphi := u_t,$$

$$\psi := v_t,$$

then system (3.1)-(3.5) can be simplified to the following initial value problem or first order evolution equation on X:

$$(AC)\left| \begin{aligned} &U_t = AU(t), \\ &U(0) = U_0 = (u_o, u_1, v_o, v_1)^T, \end{aligned} \right. \tag{3.11}$$

with

$$A := \begin{pmatrix} 0 & I & 0 & 0 \\ \Delta & -\alpha I & 0 & a(x)I \\ 0 & 0 & 0 & I \\ 0 & \alpha(x)I & \Delta & -\alpha(x)I \end{pmatrix}, \tag{3.12}$$

$$D(A) := (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \tag{3.13}$$

and $U = (u, \varphi, v, \psi)^T$.

So, we have the following result:

Theorem 3.1. *The operator A defined on* (3.12)-(3.13) *generates a* C_o *semi-group of contractions* $(S(t))_{t\geq 0}$ *in the Hilbert space X.*

Proof. Clearly $D(A)$ is dense in X. Taking the scalar product of AU and U and using the Green's identity, we have

$$\begin{aligned} < AU, U > &= < \nabla u, \nabla\varphi > \\ &\quad + < (\Delta u - \alpha(x)\varphi + \alpha(x)\psi), \varphi > \\ &\quad + < \nabla v, \nabla\psi > \\ &\quad + < (\alpha(x)\varphi + \Delta v - \alpha(x)\psi), \psi > \\ &= < \nabla u, \nabla\varphi > - < \nabla u, \nabla\varphi > \\ &\quad - < \alpha(x)\varphi, \varphi > + < \alpha(x)\varphi, \psi > \\ &\quad + < \nabla v, \nabla\psi > + < \alpha(x)\psi, \varphi > \\ &\quad - < \nabla\psi, \nabla v > - < \alpha(x)\psi, \psi > \\ &= - < \alpha(x)\varphi, \varphi > + 2 < \alpha(x)\varphi, \psi > \\ &\quad - < \alpha(x)\psi, \psi > \\ &= -\int_\Omega \alpha(x)\{\varphi^2 - 2\varphi\psi + \psi^2\}dx \\ &= -\int_\Omega \alpha(x)|\varphi - \psi|^2 dx \leq 0, \end{aligned}$$

then A is dissipative.

We claim that $0 \in \rho(A)$. In fact, we will prove that $\exists A^{-1} \in L(X)$.

Let $F = (f_1, f_2, f_3, f_4) \in X$. We will prove that there is $U \in D(A)$ such that $AU = F$, where $U = (u, \varphi, v, \psi)^T$. Thus, we have

$$\varphi = f_1, \tag{3.14}$$

$$\Delta u + \alpha(x)\{\psi - \varphi\} = f_2, \tag{3.15}$$

$$\psi = f_3, \tag{3.16}$$

$$\Delta v - \alpha(x)\{\psi - \varphi\} = f_4. \tag{3.17}$$

Then $\varphi = f_1$ and $\psi = f_3$. From (3.14) and (3.16) in (3.15), we get

$$\Delta u = \alpha(x)\{f_1 - f_3\} + f_2. \tag{3.18}$$

Then, by elliptic regularity results, exists a unique solution u in $H_0^1(\Omega) \cap H^2(\Omega)$. Now, using (3.14) and (3.16) in (3.17), we obtain

$$\Delta v = -\alpha(x)\{f_1 - f_3\} + f_4. \tag{3.19}$$

From elliptic regularity results, exists a unique solution v in $H_0^1(\Omega) \cap H^2(\Omega)$. That is, exists $U \in D(A)$ such that $AU = F$.

By other hand, if $AU = 0$, where $U = (u, \varphi, v, \psi)^T \in D(A)$, we have

$$\varphi = 0,$$

$$\Delta u + \alpha(x)\{\psi - \varphi\} = 0,$$

$$\psi = 0,$$

$$\Delta v - \alpha(x)\{\psi - \varphi\} = 0,$$

then $\varphi = 0$ and $\psi = 0$. Also, $\Delta u = 0$ and $\Delta v = 0$ imply $u = 0$ and $v = 0$. Then exists A^{-1}. Now, we prove that A^{-1} is continuous.

Multiplying (3.18) and (3.19), respectively, by u, v and integrating in Ω, we have

$$-\int_\Omega |\nabla u|^2 dx = \int_\Omega \Delta u \cdot u\, dx = \int_\Omega \alpha(x)\{f_1 - f_3\} \cdot u\, dx + \int_\Omega f_2 \cdot u dx, \tag{3.20}$$

$$-\int_\Omega |\nabla v|^2 dx = \int_\Omega \Delta v \cdot v\, dx = -\int_\Omega \alpha(x)\{f_1 - f_3\} \cdot v\, dx + \int_\Omega f_4 \cdot v\, dx. \tag{3.21}$$

From (3.20) and using Holder and Poincaré inequality, we obtain

$$0 \leq \int_\Omega |\nabla u|^2 dx = -\int_\Omega \alpha(x)\{f_1 - f_3\} \cdot u\, dx - \int_\Omega f_2 \cdot u\, dx$$

$$= \left| \int_\Omega \alpha(x)\{f_1 - f_3\} \cdot u\, dx + \int_\Omega f_2 \cdot u\, dx \right|$$

$$\leq \left| \int_\Omega \alpha(x)\{f_1 - f_3\} \cdot u\, dx \right| + \left| \int_\Omega f_2 \cdot u\, dx \right|$$

$$\leq \int_\Omega |\alpha(x)|\, |\{f_1 - f_3\}|\, |u|\, dx + \int_\Omega |f_2|\, |u|\, dx$$

$$\leq |\alpha|_\infty^{\frac{1}{2}} \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right)^{\frac{1}{2}} |u|_{L^2} + |f_2|_{L^2} |u|_{L^2}$$

$$\leq |\alpha|_\infty^{\frac{1}{2}} c_p \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right)^{\frac{1}{2}} |\nabla u|_{L^2} + c_p |f_2|_{L^2} |\nabla u|_{L^2}$$

$$\leq |\alpha|_\infty c_p^2 \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right) + \frac{1}{4} |\nabla u|_{L^2}^2 + c_p^2 |f_2|_{L^2}^2 + \frac{1}{4} |\nabla u|_{L^2}^2$$

$$= |\alpha|_\infty c_p^2 \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right) + c_p^2 |f_2|_{L^2}^2 + \frac{1}{2} |\nabla u|_{L^2}^2.$$

That is,

$$\int_\Omega |\nabla u|^2 dx \leq 2|\alpha|_\infty c_p^2 \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right) + 2c_p^2 |f_2|_{L^2}^2. \tag{3.22}$$

Analogously, from (3.21) and using Holder and Poincaré inequality, we obtain

$$\int_\Omega |\nabla v|^2 dx \leq 2|\alpha|_\infty c_p^2 \left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \right) + 2c_p^2 |f_4|_{L^2}^2. \tag{3.23}$$

Adding (3.22) with (3.23), we have

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \le \underbrace{4|\alpha|_\infty c_p^2}_{C1:=}\left(\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx\right) + 2c_p^2|f_2|_{L^2}^2 + 2c_p^2|f_4|_{L^2}^2. \tag{3.24}$$

Multiplying (3.18) and (3.19), respectively, by f_1 and f_3, we have

$$\Delta u \cdot f_1 = \alpha(x)\{f_1 - f_3\}f_1 + f_2 f_1, \tag{3.25}$$

$$\Delta v \cdot f_3 = -\alpha(x)\{f_1 - f_3\}f_3 + f_4 f_3. \tag{3.26}$$

Adding (3.25) with (3.26) and integrating in Ω, we have

$$\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx = \int_\Omega \{\Delta u \cdot f_1 + \Delta v \cdot f_3 - f_2 f_1 - f_4 f_3\} dx. \tag{3.27}$$

From (3.27), Green's identity and using Holder and Poincaré inequality, we obtain

$$\begin{aligned} 0 \le \int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx &= \int_\Omega \{-\nabla u \cdot \nabla f_1 - \nabla v \cdot \nabla f_3 - f_2 f_1 - f_4 f_3\} dx \\ &= \left|\int_\Omega \{\nabla u \cdot \nabla f_1 + \nabla v \cdot \nabla f_3 + f_2 f_1 + f_4 f_3\} dx\right| \\ &\le \left|\int_\Omega \nabla u \cdot \nabla f_1 dx\right| + \left|\int_\Omega \nabla v \cdot \nabla f_3 dx\right| \\ &\quad + \left|\int_\Omega f_2 f_1 dx\right| + \left|\int_\Omega f_4 f_3 dx\right| \\ &\le |\nabla u|_{L^2} |\nabla f_1|_{L^2} + |\nabla v|_{L^2}\, |\nabla f_3|_{L^2} + |f_2|_{L^2} |f_1|_{L^2} \\ &\quad + |f_4|_{L^2} |f_3|_{L^2} \\ &\le |\nabla u|_{L^2} |\nabla f_1|_{L^2} + |\nabla v|_{L^2} |\nabla f_3|_{L^2} + c_p |f_2|_{L^2} |\nabla f_1|_{L^2} \\ &\quad + c_p |f_4|_{L^2} |\nabla f_3|_{L^2}. \end{aligned} \tag{3.28}$$

That is,

$$C_1\int_\Omega \alpha(x)\{f_1 - f_3\}^2 dx \le |\nabla u|_{L^2} C_1|\nabla f_1|_{L^2} + |\nabla v|_{L^2} C_1|\nabla f_3|_{L^2}$$

$$+ c_p|f_2|_{L^2} C_1|\nabla f_1|_{L^2} + c_p|f_4|_{L^2} C_1|\nabla f_3|_{L^2}$$

$$\le \frac{1}{4}|\nabla u|^2_{L^2} + C_1^2|\nabla f_1|^2_{L^2} + \frac{1}{4}|\nabla v|^2_{L^2} + C_1^2|\nabla f_3|^2_{L^2}$$

$$+ \frac{1}{4}c_p^2|f_2|^2_{L^2} + C_1^2|\nabla f_1|^2_{L^2} + \frac{1}{4}c_p^2|f_4|^2_{L^2} + C_1^2|\nabla f_3|^2_{L^2}$$

$$\le \frac{1}{4}|\nabla u|^2_{L^2} + 2C_1^2|\nabla f_1|^2_{L^2} + \frac{1}{4}|\nabla v|^2_{L^2} + 2C_1^2|\nabla f_3|^2_{L^2}$$

$$+ \frac{1}{4}c_p^2|f_2|^2_{L^2} + \frac{1}{4}c_p^2|f_4|^2_{L^2}. \tag{3.29}$$

Using (3.29) in (3.24), we obtain

$$\frac{3}{4}\left\{\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\right\} \le 2C_1^2|\nabla f_1|^2_{L^2} + 2C_1^2|\nabla f_3|^2_{L^2}$$

$$+ \frac{9}{4}c_p^2|f_2|^2_{L^2} + \frac{9}{4}c_p^2|f_4|^2_{L^2}.$$

That is,

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx \le \frac{8}{3}C_1^2|\nabla f_1|^2_{L^2} + \frac{8}{3}C_1^2|\nabla f_3|^2_{L^2}$$

$$+ 3c_p^2|f_2|^2_{L^2} + 3c_p^2|f_4|^2_{L^2}. \tag{3.30}$$

Finally, we have

$$\underbrace{\int_\Omega \{|\nabla u|^2 + |\nabla v|^2 + |f_1|^2 + |f_3|^2\} dx}_{=\|U\|_X^2} \le \frac{8}{3}C_1^2|\nabla f_1|^2_{L^2} + \frac{8}{3}C_1^2|\nabla f_3|^2_{L^2}$$

$$+ 3c_p^2|f_2|^2_{L^2} + 3c_p^2|f_4|^2_{L^2}$$

$$+ c_p^2 |\nabla f_1|_{L^2}^2 + c_p^2 |\nabla f_3|_{L^2}^2$$

$$\leq C_*^2 \int_\Omega \{|\nabla f_1|^2 + |\nabla f_3|^2 + |f_2|^2 + |f_4|^2\} dx$$

$$\leq C_*^2 \|F\|_X^2.$$

That is,

$$\|A^{-1}F\|_X = \|U\|_X \leq C_* \|F\|_X.$$

Finally, using Theorem 2.1, we conclude that A is the infinitesimal generator of a C_o semi-group of contraction $(S(t))_{t\geq 0}$. And so, the abstract Cauchy problem

$$\left| \begin{array}{l} U_t = AU, \\ U(0) = U_o \in D(A), \end{array} \right.$$

has one unique solution $U(t) := S(t)U_o$. □

4. Exponential Stability

Next theorem is the main result.

Theorem 4.1. *The* C_o *semi-group of contractions* $(S(t))_{t\geq 0}$ *generated by* A, *is exponentially stable.*

Proof. We will use Theorem 2.2. First, we will prove

$$\rho(A) \supset i\mathbb{R} = \{i\beta, \beta \in \mathbb{R}\}. \tag{4.1}$$

Suppose (4.1) is false, then exists $\beta \in \mathbb{R}, \beta \neq 0$ such that $i\beta \in \sigma(A)$.

Since $0 \in \rho(A)$ and A^{-1} is compact, $\sigma(A) = \sigma_p(A)$. That is, the spectral values are eigenvalues. Then, $i\beta \in \sigma_p(A)$.

Let $U = (u, v, \psi, \varphi)^T \in D(A), U \neq 0,$ such that $(i\beta I - A)U = 0,$ i.e.,

$$AU = i\beta U. \tag{4.2}$$

Taking the scalar product of (4.2) with U, and taking its real part, we obtain

$$< AU, U >_X = i\beta\|U\|_X^2,$$

$$\mathrm{Re} < AU, U >_X = 0.$$

That is,

$$\int_\Omega \alpha(x)|\varphi - \psi|^2 dx = 0.$$

Using definition of A, (4.2) holds if and only if

$$\varphi = i\beta u, \tag{4.3}$$

$$\Delta u + \alpha(x)\{\psi - \varphi\} = i\beta\varphi, \tag{4.4}$$

$$\psi = i\beta v, \tag{4.5}$$

$$\Delta v - \alpha(x)\{\psi - \varphi\} = i\beta\psi, \tag{4.6}$$

that is, $\varphi = i\beta u$ and $\psi = i\beta v$.

Multiplying (4.4), (4.6), respectively, by φ, ψ and integrating in Ω, then using (4.3), (4.5), respectively, we obtain

$$\begin{aligned} i\beta\int_\Omega \varphi^2 dx &= \int_\Omega \Delta u\varphi\, dx + \int_\Omega \alpha(x)\{\psi - \varphi\}\varphi\, dx \\ &= \int_\Omega (\Delta u)(i\beta u)dx + \int_\Omega \alpha(x)\{\psi - \varphi\}\varphi\, dx \\ &= -i\beta\int_\Omega |\nabla u|^2 dx + \int_\Omega \alpha(x)\{\psi - \varphi\}\varphi\, dx, \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} i\beta\int_\Omega \psi^2 dx &= \int_\Omega \Delta v\psi\, dx - \int_\Omega \alpha(x)\{\psi - \varphi\}\psi\, dx \\ &= \int_\Omega (\Delta v)(i\beta v)dx - \int_\Omega \alpha(x)\{\psi - \varphi\}\psi\, dx \end{aligned}$$

$$= -i\beta\int_{\Omega}|\nabla v|^2 dx - \int_{\Omega}\alpha(x)\{\psi - \varphi\}\psi\, dx. \tag{4.8}$$

Adding (4.7) and (4.8), we have

$$i\beta\int_{\Omega}\{|\nabla u|^2 + \varphi^2 + |\nabla v|^2 + \psi^2\}dx = -\int_{\Omega}\alpha(x)\{\psi - \varphi\}^2 dx.$$

That is,

$$i\beta\|U\|^2 = -\int_{\Omega}\alpha(x)\{\psi - \varphi\}^2 dx = 0.$$

Thus $U = 0$, which is a contradiction since $U \neq 0$. Therefore, $i\mathbb{R} \subset \rho(A)$. Now, we will prove that

$$\limsup_{|\beta|\to\infty}\|(i\beta I - A)^{-1}\| < \infty. \tag{4.9}$$

Suppose (4.9) is false, i.e.,

$$\limsup_{|\beta|\to\infty}\|(i\beta I - A)^{-1}\| = \infty, \tag{4.10}$$

then exist sequences $V_m \in X$ and $\beta_m \in \mathbb{R}$ such that $\|(i\beta_m I - A)^{-1}V_m\| \geq m\|V_m\|$, $\forall m > 0$.

Thus $i\beta_m \in \rho(A)$, or equivalently $\exists(i\beta_m I - A)^{-1} \in L(X)$, that is,

$$\exists U_m \in D(A) \text{ such that } (i\beta_m I - A)U_m = V_m, \quad \|U_m\| = 1.$$

So, we have

$$U_m = (i\beta_m I - A)^{-1}V_m,$$

and

$$\|U_m\| \geq m\|\underbrace{(i\beta_m I - A)U_m}_{G_m :=}\|.$$

Then $1 = \|U_m\| \geq m\|G_m\|$, i.e., $\frac{1}{m} \geq \|G_m\|$. And taking $m \to \infty$, we get $\lim_{m\to+\infty} G_m = 0$ on X.

Now, let $U_m := (u_m, \varphi_m, v_m, \psi_m)^T$. Then

$$< G_m, U_m > = < i\beta_m U_m - AU_m, U_m >$$

$$= i\beta_m \|U_m\|^2 - < AU_m, U_m >. \tag{4.11}$$

Taking real part on inequality (4.11), we have

$$-\operatorname{Re} < AU_m, U_m > = \operatorname{Re} < G_m, U_m >,$$

and then

$$\int_\Omega \alpha(x)|\varphi_m - \psi_m|^2 dx = \operatorname{Re} < G_m, U_m > \leq \|G_m\|\|U_m\| = \|G_m\| \to 0, \tag{4.12}$$

since $< AU_m, U_m > = -\int_\Omega \alpha(x)|\varphi_m - \psi_m|^2 dx.$ Thus

$$\int_\Omega \alpha(x)|\varphi_m - \psi_m|^2 dx \to 0 \text{ when } m \to +\infty. \tag{4.13}$$

Now consider equality (4.11) and multiply by i,

$$-\beta_m\|U_m\|^2 - i \underbrace{< AU_m, U_m >}_{=\int_\Omega \alpha(x)|\varphi_m - \psi_m|^2 dx} = i\underbrace{< G_m, U_m >}_{\to 0}. \tag{4.14}$$

Since $| < G_m, U_m > | \leq \|G_m\|\|U_m\| = \|G_m\| \to 0$ and, from (4.13), $\|U_m\|^2 \to 0$ when $m \to \infty$, we get to $1 = 0$, which is a contradiction.

Thus, by Theorem 2.2, we conclude that $(S(t))_{t\geq 0}$ is exponentially stable.

We give another proof of (4.9). We know that $i\beta \in \rho(A)$, then given $F \in X$ exists $U \in D(A)$ such that

$$(i\beta I - A)U = F. \tag{4.15}$$

Denoting $F = (f_1, f_2, f_3, f_4)^T$, from (4.15), we obtain

$$i\beta u - \varphi = f_1, \tag{4.16}$$

$$i\beta\varphi - \Delta u + \alpha(x)\{\varphi - \psi\} = f_2, \tag{4.17}$$

$$i\beta v - \psi = f_3, \tag{4.18}$$

$$i\beta\psi - \Delta v - \alpha(x)\{\varphi - \psi\} = f_4. \tag{4.19}$$

Multiplying (4.16), (4.17), (4.18), (4.19), respectively, by $-\Delta u$, φ, $-\Delta v$, ψ and then integrating in Ω and using Green's identity, we obtain

$$i\beta\int_\Omega |\nabla u|^2 dx + \int_\Omega \varphi\Delta u\, dx = \int_\Omega (\nabla u)\nabla f_1 dx, \tag{4.20}$$

$$i\beta\int_\Omega \varphi^2 dx - \int_\Omega (\Delta u)\varphi\, dx + \int_\Omega \alpha(x)\{\varphi - \psi\}\varphi\, dx = \int_\Omega f_2\varphi\, dx, \tag{4.21}$$

$$i\beta\int_\Omega |\nabla v|^2 dx + \int_\Omega \psi\Delta v\, dx = \int_\Omega (\nabla f_3)\nabla v\, dx, \tag{4.22}$$

$$i\beta\int_\Omega \psi^2 dx - \int_\Omega (\Delta v)\psi\, dx - \int_\Omega \alpha(x)\{\psi - \varphi\}\psi\, dx = \int_\Omega f_4\psi\, dx. \tag{4.23}$$

Adding these four equality, we have

$$i\beta\int_\Omega \{|\nabla u|^2 + \varphi^2 + |\nabla v|^2 + \psi^2\} dx + \int_\Omega \alpha(x)|\varphi - \psi|^2 dx$$

$$= \int_\Omega (\nabla u)\nabla f_1 dx + \int_\Omega \varphi f_2 dx + \int_\Omega (\nabla v)\nabla f_3 dx + \int_\Omega \psi f_4 dx.$$

That is,

$$i\beta\|U\|_X^2 + \int_\Omega \alpha(x)|\varphi - \psi|^2 dx = < U, F >_X .$$

Taking its imaginary part and using Cauchy-Schwartz inequality, we have

$$\beta\|U\|_X^2 = \mathrm{Im} < U, F >_X \le \|U\|_X \|F\|_X.$$

Then

$$\beta \|U\|_X \leq \|F\|_X,$$

that is,

$$\|(i\beta I - A)^{-1} F\| = \|U\|_X \leq \frac{1}{\beta} \|F\|_X,$$

i.e.,

$$\|(i\beta I - A)^{-1}\| \leq \frac{1}{\beta}.$$

□

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